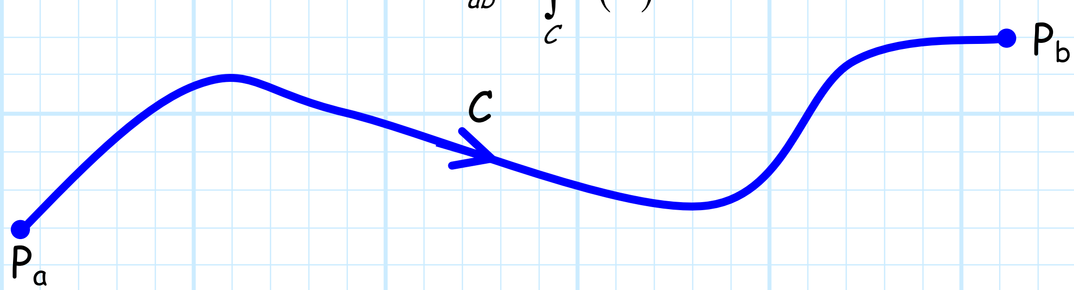
Voltage and Electric Potential

An important application of the line integral is the calculation of **work**. Say there is some vector field $\mathbf{F}(\bar{r})$ that exerts a **force** on some object.

Q: *How much **work** (W) is done by this vector field if the object moves from point P_a to P_b, along contour C ??*

A: We can find out by evaluating the line integral:

$$W_{ab} = \int_C \mathbf{F}(\bar{r}) \cdot \overline{d\ell}$$

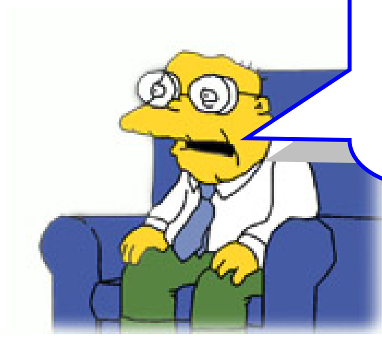


Say this object is a **charged particle** with charge Q, and the force is applied by a static **electric field** $\mathbf{E}(\bar{r})$. We **know** the force on the charged particle is:

$$\mathbf{F}(\bar{r}) = Q\,\mathbf{E}(\bar{r})$$

and thus the work done **by the electric field** in moving a charged particle along some contour C is:

$$W_{ab} = \int_C \mathbf{F}(\bar{\mathbf{r}}) \cdot \overline{d\ell} = Q \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell}$$



Q: *Oooh, I don't like evaluating contour integrals; isn't there some* ***easier*** *way?*

A: **Yes** there is! Recall that a **static** electric field is a **conservative** vector field. Therefore, we can write any electric field as the **gradient** of a specific **scalar** field $V(\bar{\mathbf{r}})$:

$$\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{\mathbf{r}})$$

We can then evaluate the work integral as:

$$\begin{aligned} W_{ab} &= Q \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} \\ &= -Q \int_C \nabla V(\bar{\mathbf{r}}) \cdot \overline{d\ell} \\ &= -Q \left[V(\bar{\mathbf{r}}_b) - V(\bar{\mathbf{r}}_a) \right] \\ &= Q \left[V(\bar{\mathbf{r}}_a) - V(\bar{\mathbf{r}}_b) \right] \end{aligned}$$

We **define**:

$$V_{ab} \doteq V(\bar{r}_a) - V(\bar{r}_b)$$

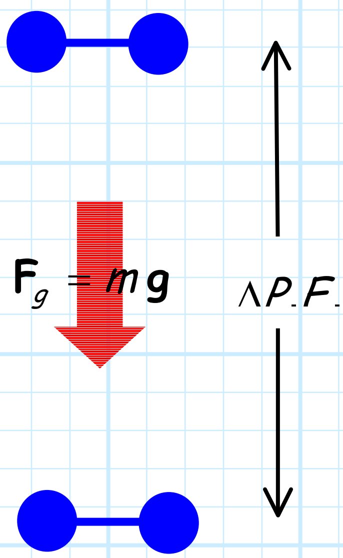
Therefore:

$$W_{ab} = Q\, V_{ab}$$

Q: *So what the heck **is** V_{ab} ? Does it mean any thing? Do we use it in engineering?*

A: First, consider what W_{ab} is!

The value W_{ab} represents the work done **by** the electric field **on** charge Q when moving it from point P_a to point P_b. This is **precisely** the same concept as when a **gravitational force** field moves an object from one point to another.



The work done by the gravitational field in this case is equal to the **difference** in **potential energy** (*P.E.*) between the object at these two points.

The value W_{ab} represents the **same** thing! It is the **difference** in **potential energy** between the charge at point P_a and at P_b.

Q: *Great, now we know what W_{ab} is. But the question was, **WHAT IS** V_{ab} !?!*

A: That's easy! Just rearrange the above equation:

$$V_{ab} = \frac{W_{ab}}{Q}$$

See? The value V_{ab} is equal to the difference in potential energy, **per coulomb of charge**!

* In other words V_{ab} represents the difference in potential energy for **each** coulomb of charge in Q.

* Another way to look at it: V_{ab} is the difference in potential energy if the particle has a charge of **1 Coulomb** (i.e., $Q=1$).

Note that V_{ab} can be expressed as:

$$\begin{aligned} V_{ab} &= \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} \\ &= V(\bar{r}_a) - V(\bar{r}_b) \end{aligned}$$

where point P_a lies at the **beginning** of contour C, and P_b lies at the **end**.

We refer to the **scalar field** $V(\bar{r})$ as the **electric potential function**, or the electric potential **field**.

We likewise refer to the scalar value V_{ab} as the electric potential **difference**, or simply the **potential difference** between point P_a and point P_b.

Note that V_{ab} (and therefore $V(\bar{r})$), has units of:

$$V_{ab} = \frac{W_{ab}}{Q} \quad \left[\frac{\text{Joules}}{\text{Coulomb}}\right]$$

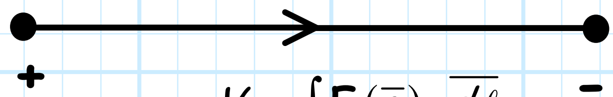
Joules/Coulomb is a rather **awkward** unit, so we will use the other name for it—**VOLTS**!

$$\frac{1 \text{ Joule}}{\text{Coulomb}} \doteq 1 \text{ Volt}$$

Q: *Hey! We used volts in **circuits** class. Is this the **same** thing?*

A: It is **precisely** the same thing!

Perhaps this will help. Say P_a and P_b are two points somewhere on a circuit. But let's call these points something different, say point + and point - .



$$V = \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell}$$

Therefore, V represents the **potential difference** (in volts) **between** point (i.e., **node**) + and point (**node**) - . Note this value can be either **positive** or **negative**.

Q: *But, does this mean that circuits produce **electric fields**?*

A: **Absolutely**! Anytime you can measure a **voltage** (i.e., a potential difference) between two points, an electric field **must** be present!